Gaussian unitary ensemble statistics in a time-reversal invariant microwave triangular billiard

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(Received 27 June 2000)

The spectrum of a chaotic two-dimensional quantum billiard with threefold symmetry has been studied in an experiment with a superconducting microwave cavity. In total 622 eigenvalues were identified experimentally and compared with numerical calculations. The statistical analysis of the data shows that Gaussian unitary ensemble statistics can be observed for a spectrum of a time-reversal invariant system.

PACS number(s): 05.45.Mt, 41.20.Jb, 02.60.Cb, 11.30.Er

For the quantum counterpart of a classical chaotic system the statistical distribution of the energy-eigenvalues is known to be well described by random matrix theory [1–3]. Whether the statistics follows the Gaussian orthogonal ensemble (GOE), the Gaussian unitary ensemble (GUE), or the Gaussian symplectic ensemble (GSE) depends on the symmetries of the system [4]. While GUE statistics usually is expected for spectra of non-time-reversal invariant systems, time-reversal invariant systems with integral spin or rotational invariance should show level-distributions according to GOE. It has been shown that this is not true in general and systems with certain discrete symmetries may have spectra with GUE statistics although they are time-reversal invariant [5]. The most simple example for such a system is a chaotic two-dimensional billiard with threefold symmetry. This so-called triangular billiard has a shape that is invariant under rotations of 120°, while the geometry has no further symmetries, e.g., mirror-symmetries. For a triangular billiard there are three classes of eigenfunctions to the Hamiltonian H that have different symmetry properties [5,6]. These classes of eigenfunctions can be classified by an integer quantum number l which describes the transformation of the wave function $\Psi^{(l)}$ under a rotation R of 120°:

$$H\Psi_n^{(l)}=E_n\Psi_n^{(l)}, \tag{1}$$

$$R\Psi_n^{(l)}=\exp\left(i\frac{2\pi}{3}l\right)\Psi_n^{(l)}. \tag{2}$$

For $l=0$ the wave functions are invariant under time-reversal T and the corresponding field patterns have threefold symmetry [Eq. (2)]. As T is given by complex conjugation, these wave functions are real. The other two classes ($l=\pm1$) have non-time-reversal invariant wave functions, which are therefore complex, and the field patterns have no threefold symmetry [Eq. (2)]. Under time-reversal the eigenfunctions of these two classes are interchanged so that for every mode with $l=+1$ there exists a time-reversed mode with $l=-1$ in the other class and vice versa. Both wave functions $\Psi^{(+1)}$ and $\Psi^{(-1)}=T\Psi^{(+1)}=(\Psi^{(+1)})^*$ are eigenfunctions to the same eigenvalue. Therefore, there is no contradiction to the time-reversal invariance of the triangular billiard, because a complete set of real eigenfunctions always can be constructed by linear combination of the eigenfunctions to the same eigenvalue. Due to the degeneracy of the levels corresponding to the two non-time-reversal invariant classes of wave functions, a triangular billiard's spectrum shows a characteristic singlet-doublet structure. If there is a small perturbation which causes a splitting of the doublets, it is possible to distinguish the eigenvalues of time-reversal invariant and non-time-reversal invariant eigenmodes. In [5] GUE statistics for the doublets and GOE statistics for the singlets is predicted. To study those effects it was necessary to construct a fully chaotic triangular billiard. Numerical simulations of the motion in a classical billiard were used to find suitable geometries without bouncing ball orbits. We chose one particular shape (see Fig. 1) which can easily be parametrized in polar coordinates by

$$r(\varphi)=r_0[1+0.2\cos(3\varphi)-0.2\sin(6\varphi)]. \tag{3}$$

The experimental technique used here is to measure the eigenfrequencies of a microwave cavity. In general the electric field $\vec{E}(\vec{r},t)=\Sigma_\omega\vec{E}_\omega(\vec{r})e^{-i\omega t}$ inside a cavity is described by the vectorial Helmholtz equation

$$\left(\Delta+\frac{\omega^2}{c^2}\right)\vec{E}_\omega(\vec{r})=0, \tag{4}$$

with the requirement that $\vec{E}$ tangential vanishes at the boundary. This implies the assumption that the cavity's walls are ideally conducting. In a flat cavity with cylindrical symmetry, whose extension in z direction is small compared to the

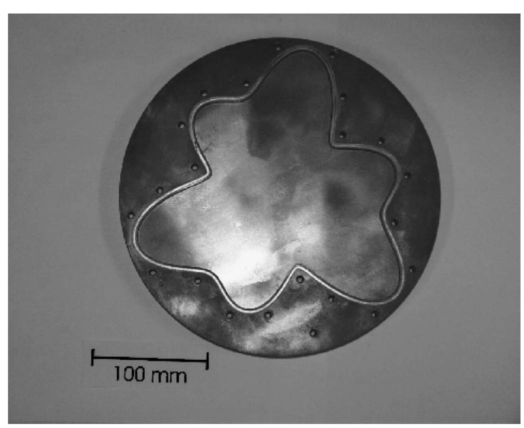

FIG. 1. Photograph of the microwave cavity without its lid.

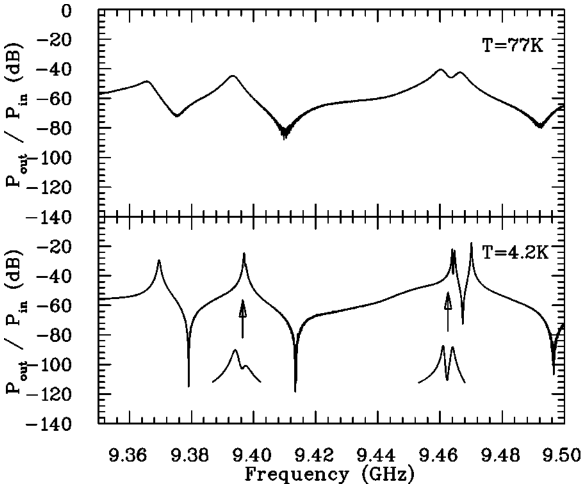

FIG. 2. Part of a spectrum measured at $T=77$ K and $T=4.2$ K. The superconducting cavity at $T=4.2$ K has a higher quality factor and the spectrum hence shows an improved signal-to-noise ratio.

wavelength of the electric field, the wave propagation becomes two-dimensional and the Helmholtz equation reduces to the scalar wave equation

$$\left(\Delta+\frac{\omega^2}{c^2}\right)\phi_\omega(x,y)=0, \tag{5}$$

with Dirichlet boundary conditions [7], which is equivalent to the time-independent Schrödinger equation for the quantum billiard. There are only transverse-magnetic (TM) modes with the electric field vectors $\vec{E}_\omega(\vec{r})=\phi_\omega(x,y)\vec{e}_z$. This means that experiments (see, e.g., [8–11]) where the eigenmodes of such a cavity are determined serve as analog computers that solve the stationary Schrödinger equation for a quantum billiard of the same shape. In the present experiment, as in previous ones at Darmstadt (see, e.g., [9,10,12,13]), a superconducting cavity is used. The quality factor $Q=f/\Delta f$, where Δf is the width of a resonance at the frequency f, of $10^4,\ldots,10^5$ is larger than in the normal conducting case ($Q\approx 10^3$) and therefore allows a better identification of resonances in the spectrum (Fig. 2). The cavity resonator (Fig. 1) consists of a bottom, an inset, and a lid. To ensure proper electric contact at high frequencies, wires of solder are placed between the parts of the billiard, which are then sandwiched between two steel plates. All parts are squeezed by screws. The billiard itself is made from copper and has a lead/tin surface, which becomes superconducting at temperatures below 7.2 K. In the experiment the resonator is cooled to 4.2 K in a cryostat with liquid helium, where it remains at constant temperature and pressure for several days. The microwaves are coupled into the cavity by dipole antennas, which are connected to a RF source by microwave cables. There are four antennas, which reach into the billiard through small holes in the lid. The reflected power for each antenna and the transmitted power for each antenna combination is measured for the frequency range from 1.25 to 25 GHz in steps of 10 kHz, so that ten different spectra are obtained. The signals are analyzed by a HP-8510C network analyzer. The network analyzer is controlled by a personal computer, which also stores the data. The experimental spectrum shows the expected singlet-doublet structure (Fig. 2). This means that the mechanical imperfections of the cavity and the perturbative influences of the antennas, which are always present in an experiment and even necessary here, cause doublet splittings which can be resolved within the frequency resolution of our setup using a superconducting microwave cavity with a high quality factor. Typical splittings are in the range from 0.5 to 3.0 MHz. For the investigated spectrum from 1.25 to 25.0 GHz there is no evidence for a systematic dependence of those splittings on frequency. The values of the splittings simply scatter around the mean. Altogether 622 levels up to a frequency of 25 GHz have been identified. This is in good agreement with the number of modes expected from the dimensions of the billiard and indicates that in principle each doublet splits up into two individual resonances. For frequencies higher than approximately 25.5 GHz the wave propagation inside the cavity gets three-dimensional and no longer serves as a simulation of a quantum billiard. In order to separate singlet and doublet modes correctly the distance of the two following resonances is considered. If it is bigger than a maximum value, the two resonances are not regarded as parts of one doublet. If there are three resonances close together the pair with the smaller distance is counted as a doublet. By taking a value of 8 MHz for the maximum splitting the levels are divided into 196 singlets and 213 doublets. As Fig. 3 indicates the maximum splitting is not a critical parameter, because the number of doublets almost saturates for values larger than 6 MHz. The influence of the assumed maximum splitting will be discussed later on. Each level sequence $\{f_i\}$ of singlets and doublets defines a level density function $\rho(f)=\Sigma_i\delta(f-f_i)$ for the spectrum. By integration a staircase function

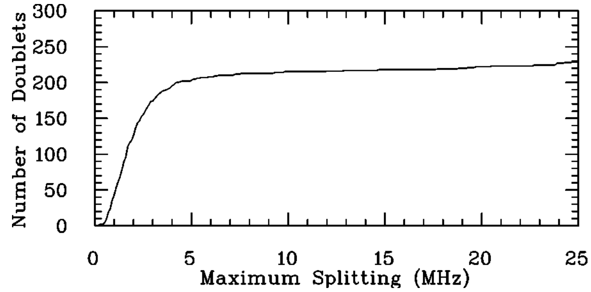

FIG. 3. Total number of doublets depending on the value taken for the maximum splitting.

$$N(f)=\int_0^f\rho(f')df'=N_{smooth}(f)+N_{fluct}(f), \tag{6}$$

which consists of a smooth and a fluctuating part, follows. The smooth part is given by the Weyl formula

$$N_{Weyl}(f)=v_1f^2+v_2f+v_3 \tag{7}$$

for two-dimensional systems and carries no information on the dynamics of the system. It describes the mean behavior which only depends on area and perimeter of the billiard cavity [14], while the fluctuating part contains the relevant information. The fluctuating and the smooth part are separated by fitting the Weyl formula to the staircase function with parameters v_1, v_2, and v_3. For further statistical analy-

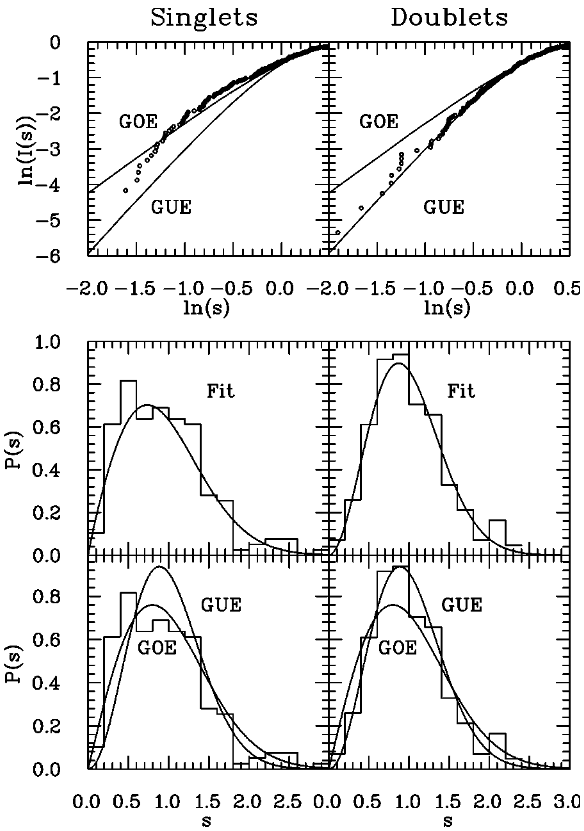

FIG. 4. Nearest-neighbor-spacings distributions (bottom) and integrated NNDs (top) for singlets and doublets.

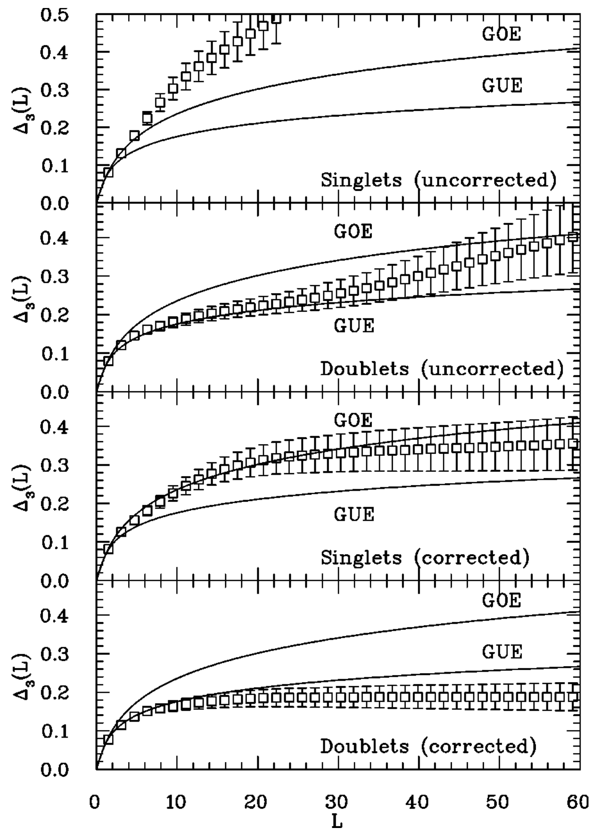

FIG. 5. Dyson-Mehta statistics for the raw spectra (upper part) and for the corrected spectra (lower part) showing GOE statistics for the singlets and GUE statistics for the doublets.

sis and comparison with theoretical predictions the spectra have to be unfolded. By using the transformation $\epsilon_i = N_{Weyl}(f_i)$ this procedure leads to spectra $\{\epsilon_i\}$, with constant mean level densities and mean level spacings normalized to 1. First a nearest-neighbor distribution (NND) is considered. It shows the distribution $P(s)$ of the spacings $s_i = \epsilon_{i+1} - \epsilon_i$ between two neighboring levels. Both GOE and GUE statistics imply level repulsion and show a maximum of the corresponding spacings distributions at the value of the mean level spacing. This can be seen from the NNDs for the singlet and the doublet spectra (Fig. 4). A different behavior of singlets and doublets according to GOE and GUE can be recognized, although it is difficult to distinguish between GOE and GUE statistics by eye. By fitting the curve $P(s) = c(a) s^a \exp[-b(a)s^2]$, which generalizes the spacings distributions of GOE and GUE, with one fit parameter a (b and c depend on a and are fixed due to normalization) to the NNDs the different statistical properties of singlets and doublets can be seen more clearly (Fig. 4). The fit gives $a = 0.90 \pm 0.18$ for the singlets and $a = 1.88 \pm 0.09$ for the doublets, to be compared with $a = 1.00$ and $a = 2.00$ for GOE and GUE, respectively. A sometimes more sensitive analysis of the so-called short range correlations of the spectra is the integrated NND $I(s) = \int P(s) ds$. In a logarithmic plot (upper part of Fig. 4) small deviations from random matrix theory predictions appear. To discuss the long-range correlations of the spectra, the Dyson-Mehta statistics [2,15] is considered. As can be seen from the upper half of Fig. 5, the doublets follow GUE up to $L \approx 30$ and show an increase of the Δ_3 statistics for larger values of L. Such an increase can also be seen for the singlets, which follow GOE up to $L \approx 8$. Then nonuniversal behavior sets in. Furthermore there is no saturation behavior. When varying the parameter taken for the maximum splitting of the doublets, this behavior does not disappear. To get a deeper insight, the two sets of eigenvalues have been compared with a numerical simulation for the corresponding quantum billiard entirely in the spirit of [16]. The comparison shows that about 600 calculated levels can clearly be identified in the experimental spectra. It also can be seen that for some frequency ranges the quality factor of the cavity still is not high enough to resolve all resonances. Therefore some triplets, which occur when a singlet and a doublet are close together, cannot be resolved as three resonances. There are some missing and extra modes, too. In particular, it is sometimes not possible to distinguish between singlets and doublets, and pairs of singlets appear which are taken for doublets because of a small distance. Furthermore, some doublets show no splitting. With the help of the calculation, however, the few necessary corrections to the measured set of eigenfrequencies can easily be made. Finally a total number of 623 levels results, with 207 singlets and 208 doublets. The Dyson-Mehta statistics shows perfect agreement with theory when the results of the comparison are taken into account (Fig. 5, lower half). This reflects the sensitivity of the long-range correlations on missing or misinterpreted levels, which usually do not effect the short-range correlations (see, e.g., [17]).

The anomalous spectral statistics of a chaotic triangular billiard was studied by an experiment with a superconducting microwave cavity providing the necessary frequency resolution to resolve doublet splittings, which allowed us to separate the eigenvalues of states with different quantum numbers. The measured spectrum contains approximately the same number of singlet and doublet modes. About 95% of the resonances were identified clearly, and a comparison with a numerical calculation allowed us to fix the remaining 5% of problematic cases. As a result, GUE-like statistics of a time-reversal invariant system's spectrum was observed experimentally. The discussed experiment gives an experimental proof of the predictions developed in [5]. It should be mentioned here that another experiment was done with a cavity with threefold symmetry and extra mirror symmetries. This cavity is presently being investigated and the preliminary data seem to indicate that the doublets show GOE statistics because of the different symmetry properties. This is in agreement with [5], as well.

We would like to thank O. Bohigas and T. Seligman for various hints and helpful discussions and the Deutsche Forschungsgemeinschaft (DFG) for supporting this work under Contract No. Ri 242/16-1.

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